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Large N gauge theories and topological cigars

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Abstract: We analyze the conjectured duality between a class of double-scaling limits of a one-matrix model and the topological twist of non-critical superstring backgrounds that contain the $N = 2$ Kazama-Suzuki $SL(2)/U(1)$ supercoset model. The untwisted backgrounds are holographically dual to double-scaled Little String Theories in four dimensions and to the large N double-scaling limit of certain supersymmetric gauge theories. The matrix model in question is the auxiliary Dijkgraaf-Vafa matrix model that encodes the F-terms of the above supersymmetric gauge theories. We evaluate matrix model loop functions with the goal of extracting information on the spectrum of operators in the dual non-critical bosonic string. The twisted coset at level one, the topological cigar, is known to be equivalent to the $c = 1$ non-critical string at selfdual radius and to the topological theory on a deformed conifold. In this case, we find that the matrix model double-scaling limit describes a subset of the full spectrum of the $c = 1$ theory at selfdual radius. In particular, it contains operators that do not satisfy the Seiberg bound.

Keywords: Matrix Models, Topological Strings, 1/N Expansion.

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Contents

1. Introduction

Since the work of 't Hooft [1], which suggests that the large N limit of four-dimensional $SU(N)$ gauge theory admits a weakly coupled string theory description, there has been considerable interest in trying to find concrete examples of such a duality, with the hope of gaining insight and analytical control over non-perturbative phenomena like confinement and chiral symmetry breaking.

The AdS/CFT correspondence [2] and its generalizations are examples of such gauge/string dualities. On the string side, however, one is usually limited to the supergravity approximation due to technical difficulties of dealing with Ramond-Ramond fluxes. Consequently, it is interesting to study examples where one has a better control over the dual string worldsheet theory and some of these hurdles can be overcome.

In [3, 4], the large N limit of a wide class of four-dimensional $\mathcal{N} = 1$ theories in a partially confining phase was studied. It was found that the low-energy description of the theory breaks down close to points in the parameter space where some baryonic and/or mesonic states become massless. Nevertheless, this can be cured by defining a large N

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double-scaling limit (DSL) where one approaches the singularity by keeping the mass M of these states fixed. This limit has several interesting features. For example, the conventional 't Hooft limit leads to a free theory of colour singlet states where all interactions are suppressed by powers of $1/N$. In the present case, the large N Hilbert space splits into two decoupled sectors, and one of them keeps residual interactions whose strength is inversely proportional to the mass M. This suggests that the dynamics of this sector has a dual string description where the string coupling is given by $1/N_{eff} \sim \sqrt{T}/M$ where T is the tension of the confining string. For some of these models where supersymmetry is actually enhanced in the DSL from $\mathcal{N}=1$ to $\mathcal{N}=2$ or even $\mathcal{N}=4$, the dynamics of the interacting subsector can be described by a double-scaled Little String Theory or, via holography, by a non-critical superstring background with no Ramond-Ramond flux and an exactly solvable worldsheet theory [8, 9, 5, 10, 6, 7].

The above large N duality proposals are based on an analysis of the F-terms of the theory, which, following the results of Dijkgraaf and Vafa, is performed by means of an auxiliary matrix model [11 − 13]. In [4], it was shown that these large N DSLs correspond to a double-scaling limit of the auxiliary matrix model that is analogous to the double-scaling limits considered in [14] to study $c \leq 1$ systems coupled to 2d gravity. In particular, it was shown that the double-scaling limits are well-defined in higher genus as well and that the free energy of the matrix model scales as

$$F_g \sim M^{2-2g} \ . \tag{1.1}$$

Furthermore, it can be argued [15] on the basis of the Dijkgraaf-Vafa correspondence and previous studies [19, 18, 16, 17, 8, 9] that the $c \leq 1$ system defined by the matrix model DSL considered in [3] is dual to the topologically twisted version of the non-critical superstring backgrounds.

In this paper, we pursue the study of the matrix models DSLs introduced in [3], focusing on those models where the large N double-scaled theory has $\mathcal{N}=2$ supersymmetry. The goal is to verify the duality between the matrix model DSLs and the topologically twisted non-critical superstring backgrounds. This would be a consistency check of the holographic duality between Little String Theories defined in the proximity of Calabi-Yau singularities and non-critical superstring backgrounds proposed in [8, 9]. It would also establish a more direct link between the large N DSLs of [3] and the non-critical superstring backgrounds, at least at the level of the F-terms.

To achieve this, in section 3, we evaluate loop functions of the double-scaled matrix model in the planar limit. The rationale here is that from these matrix model loop functions one can extract genus zero correlation functions of local operators in the dual $c \leq 1$ non-critical bosonic theory, as was done in [22] in the case of minimal models coupled to 2d gravity (see also [20, 21] and [23] for a comprehensive review). The tool we use is the algorithm developed in [24] to solve the matrix model loop equations. This is a particularly useful technique because the nature of the singularities is such that the orthogonal polynomial technique cannot be applied [4].

This is a preliminary step towards verifying that the $c \leq 1$ non-critical bosonic string indeed corresponds to the topologically twisted non-critical superstring background. The

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simplest DSL that can be defined is associated to a conifold singularity and the relative non-critical superstring background is the $\mathcal{N} = 2$ supersymmetric coset $SL(2)_k/U(1)$ at level $k = 1$ [18, 16, 17]. In this case, the matrix model DSL is expected to capture the topological A-twist of this background (the "topological cigar"). The topological cigar was shown to be equivalent to the $c = 1$ non-critical bosonic theory at selfdual radius [16] (see [25 − 27] for a recent analysis), which is equivalent to the topological B model on the deformed conifold [17].

We will show that in this case the matrix model DSL describes a subset of the full spectrum of the $c = 1$ theory at selfdual radius. In particular, it contains operators that do not satisfy the Seiberg bound. Therefore, at least for the case of the conifold singularity, we see an explicit relation between the DSL of the Dijkgraaf-Vafa matrix model and the topologically twisted non-critical superstring backgrounds.

The large N DSLs of [3] are defined in the neighbourhood of Argyres-Douglas-type singularities. In section 4, we will compare the double-scaled free energy of the matrix model at genus zero and the prepotential of the relevant $\mathcal{N} = 2$ Seiberg-Witten curve. In particular, we will evaluate their third derivatives with respect to the glueball superfields and the $\mathcal{N} = 2$ moduli and find precise agreement upon rescaling as one approaches the singularity. This is a further check of the duality in the planar limit. We will also suggest a precise relation between the higher genus terms of the double-scaled matrix model and the higher genus terms of the $\mathcal{N} = 2$ Seiberg-Witten free energy in a neighbourhood of the Argyres-Douglas singularity. The Seiberg-Witten partition function in the neighbourhood of an Argyres-Douglas singularity would then correspond to the matrix model partition function defined by the near-critical spectral curve, which makes contact with a conjecture by Nekrasov [61].

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2. The matrix model double-scaling limit

In this section, we will review the matrix model singularities and relative double-scaling limits studied in [3, 4]. Consider an $\mathcal{N} = 1$ $U(N)$ theory with a chiral adjoint field Φ and superpotential $W(\Phi)$. The classical vacua of the theory are determined by the stationary points of $W(\Phi)$

$$W(\Phi) = N \sum_{i=1}^{\ell+1} \frac{g_i}{i} \mathrm{Tr}_N\, \Phi^i \ . \tag{2.1}$$

The overall factor N ensures that the superpotential scales appropriately in the 't Hooft limit. For generic values of the couplings, we find ℓ stationary points at the zeroes of

$$W'(x) = N\varepsilon \prod_{i=1}^{\ell} (x - a_i) \ , \qquad \varepsilon \equiv g_{\ell+1} \ . \tag{2.2}$$

The classical vacua correspond to configurations where each of the N eigenvalues of Φ takes one of the ℓ values, $\{a_i\}$, for $i = 1, \ldots, \ell$. Thus vacua are related to partitions of N where $N_i \geq 0$ eigenvalues take the value a_i with $N_1 + N_2 + \ldots N_\ell = N$. Provided $N_i \geq 2$ for all

i, the classical low-energy gauge group in such a vacuum is

$$\hat{G}_{cl} = \prod_{i=1}^{\ell} U(N_i) \approx \prod_{i=1}^{\ell} U(1)_i \times SU(N_i) \ . \tag{2.3}$$

Strong-coupling dynamics will produce non-zero gluino condensates in each non-abelian factor of $\hat{G}_{cl}$. If we define as $W_{\alpha i}$ the chiral field strength of the $SU(N_i)$ vector multiplet in the low-energy theory, we can define a corresponding low-energy glueball superfield $S_i = -(1/32\pi^2)\langle Tr_{N_i}(W_{\alpha i}W^{\alpha i})\rangle$ in each factor. Non-perturbative effects generate a superpotential of the form [28 – 30]

$$W_{eff}(S_1, \ldots, S_\ell) = \sum_{j=1}^{\ell} N_j(S_j \log(\Lambda_j^3/S_j) + S_j) + 2\pi i \sum_{j=1}^{\ell} b_j S_j \ , \tag{2.4}$$

where the b_j are integers defined modulo N_j that label inequivalent supersymmetric vacua.

Dijkgraaf and Vafa argued that the exact superpotential of the theory can be determined by considering a matrix model with potential $W(\hat{\Phi})$ [11, 12]

$$\int d\hat{\Phi} \ \exp\left(-g_s^{-1} \ Tr\, W(\hat{\Phi})\right) = \exp \sum_{g=0}^{\infty} F_g \ g_s^{2g-2} \, , \tag{2.5}$$

where $\hat{\Phi}$ is an $\hat{N} \times \hat{N}$ matrix in the limit $\hat{N} \to \infty$. The integral has to be understood as a saddle-point expansion around a critical point where $\hat{N}_i$ of the eigenvalues sit in the critical point a_i. Note that $\hat{N}$ is not related to the N from the field theory. The glueball superfields are identified with the quantities

$$S_i = g_s \hat{N}_i \, , \qquad S = \sum_{i=1}^{\ell} S_i = g_s \hat{N} \tag{2.6}$$

in the matrix model and the exact glueball superpotential is

$$W_{gb}(S_1, \ldots, S_\ell) = \sum_{j=1}^{\ell} N_j \frac{\partial F_0}{\partial S_j} + 2\pi i \sum_{j=1}^{\ell} b_j S_j \, , \tag{2.7}$$

where F_0 is the genus zero free energy of the matrix model in the planar limit.

The central object in matrix model theory is the resolvent

$$\omega(x) = \frac{1}{\hat{N}} \mathrm{Tr} \left(\frac{1}{x - \hat{\Phi}} \right) \ . \tag{2.8}$$

At leading order in the $1/\hat{N}$ expansion, $\omega(x)$ is valued on the spectral curve Σ, in this case a hyper-elliptic Riemann surface defined by the algebraic relation

$$y^2 = \frac{1}{(N\varepsilon)^2} \left(W'(x)^2 + f_{\ell-1}(x) \right) \ . \tag{2.9}$$

The numerical prefactor is chosen for convenience. In terms of this curve

$$\omega(x) = W'(x) - N\varepsilon y(x) \ . \tag{2.10}$$

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In (2.9), $f_{\ell-1}(x)$ is a polynomial of order $\ell - 1$ whose ℓ coefficients are moduli that are determined by the S_i. In general, the spectral curve can be viewed as a double-cover of the complex plane connected by ℓ cuts. For the saddle-point of interest only s of the cuts may be opened and so only s of the moduli $f_{\ell-1}(x)$ can vary. Consequently $y(x)$ has $2s$ branch points and $\ell - s$ zeros:[1]

$$\Sigma: \qquad y^2 = Z_m(x)^2 \sigma_{2s}(x) \tag{2.11}$$

where $\ell = m + s$ and

$$Z_m(x) = \prod_{j=1}^{m}(x - z_j)\,, \qquad \sigma_{2s}(x) = \prod_{j=1}^{2s}(x - \sigma_j)\ . \tag{2.12}$$

The remaining moduli are related to the s parameters $\{S_i\}$ by (2.6)

$$S_i = g_s \hat{N}_i = N\varepsilon \oint_{A_i} y\, dx\,, \tag{2.13}$$

where the cycle A_i encircles the cut which opens out around the critical point a_i of $W(x)$.

Experience with the "old" matrix model teaches us that double-scaling limits can exist when the parameters in the potential are varied in such a way that combinations of branch and double points come together. In the neighbourhood of such a critical point,[2]

$$y^2 \longrightarrow C Z_m(x)^2 B_n(x)\,, \tag{2.14}$$

where $z_j, b_i \to x_0$, which we can take, without loss of generality, to be $x_0 = 0$. The double-scaling limit involves first taking $a \to 0$

$$x = a\tilde{x}\,, \qquad z_i = a\tilde{z}_i\,, \qquad b_j = a\tilde{b}_j \tag{2.15}$$

while keeping tilded quantities fixed. In the limit, we can define the near-critical curve Σ_-:[3]

$$\Sigma_- : \qquad y_-^2 = \tilde{Z}_m(\tilde{x})^2 \tilde{B}_n(\tilde{x})\ . \tag{2.16}$$

It was shown in [4], generalizing a result of [3], that in the limit $a \to 0$, in its sense as a complex manifold, the curve Σ factorizes as $\Sigma_- \cup \Sigma_+$. The complement to the near-critical curve is of the form

$$\Sigma_+ : \qquad y_+^2 = x^{2m+n} F_{2s-n}(x)\ . \tag{2.17}$$

where $F_{2s-n}(x)$ is regular when $a = 0$.

[1] Occasionally, for clarity, we indicate the order of a polynomial by a subscript.

[2] We have chosen for convenience to take all the double zeros $\{z_j\}$ into the critical region.

[3] For polynomials, we use the notation $\tilde{f}(\tilde{x}) = \prod_i(\tilde{x} - \tilde{f}_i)$, where $f(x) = \prod_i(x - f_i)$, $x = a\tilde{x}$ and $f_i = a\tilde{f}_i$.

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2.1 Engineering the double-scaling limit on-shell

It is important to stress that the above singularities are obtained on shell [3, 4]. In the context of supersymmetric gauge theories, the moduli $\{S_i\}$ are fixed by extremizing the glueball superpotential (2.7). It is not, *a priori*, clear whether a double-scaling limit can be reached whilst simultaneously being on-shell with respect to the glueball superpotential. We will now review the analysis of [3, 4] and show that suitable choices of the coupling constants $\{g_i\}$ do indeed allow for a double-scaling limit on-shell with respect to the glueball superpotential. In general, the potentials required are non-minimal. However, this is irrelevant for extracting the universal behaviour that only depends on the near-critical curve (2.14).

So the problem before us is to show that the critical point can be reached simultaneously with being at a critical point of the glueball superpotential. It is rather difficult to find the critical points of the latter directly. Fortunately another more tractable method consists of comparing the matrix model spectral curve (2.9), the "$\mathcal{N}=1$ curve", with the Seiberg-Witten curve of the underlying $\mathcal{N}=2$ theory that results when the potential vanishes. The latter has the form

$$y_{\rm SW}^2 = P_N(x)^2 - 4\Lambda^{2N}\,, \tag{2.18}$$

where $P_N(x) = \prod_{i=1}^N (x-\phi_i)$. Here, $\{\phi_i\}$ are a set of coordinates on the Coulomb branch of the $\mathcal{N}=2$ theory and Λ is the usual scale of strong-coupling effects in the $\mathcal{N}=2$ theory.

When the $\mathcal{N}=2$ theory is deformed by addition of the superpotential (2.1), it can be shown that a vacuum exists when the Seiberg-Witten curve and the $\mathcal{N}=1$ curve represent the same underlying Riemann surface [28, 31, 32]. In concrete terms this means that, on-shell,

$$\begin{aligned} y_{\rm SW}^2 &= P_N(x)^2 - 4\Lambda^{2N} = H_{N-s}(x)^2\sigma_{2s}(x) \\ y^2 &= \frac{1}{(N\varepsilon)^2}\big(W'(x)^2 + f_{s+m-1}(x)\big) = Z_m(x)^2\sigma_{2s}(x)\,, \end{aligned} \tag{2.19}$$

In these equations, $H_{N-s}(x)$, $\sigma_{2s}(x)$, $Z_m(x)$ are polynomials of the indicated order, and we choose (in order to remove some redundancies)

$$H_{N-s}(x) = x^{N-s} + \cdots\,, \qquad \sigma_{2s}(x) = x^{2s} + \cdots\,, \qquad Z_m(x) = x^m + \cdots\,. \tag{2.20}$$

Both curves describe the same underlying Riemann surface, namely the reduced curve of genus $s-1$ which is a hyper-elliptic double-cover of the complex plane with s cuts. All-in-all there are $2(N+l)$ equations for the same number of unknowns in $\{P, H, \sigma, Z, f\}$. There are many solutions to these equations and we can make contact with the description of the vacua in section 1 by taking the classical limit $\Lambda \to 0$; whence

$$P_N(x) \to \prod_{i=1}^{\ell}(x-a_i)^{N_i}\,, \qquad \sum_{i=1}^{\ell} N_i = N\,, \tag{2.21}$$

so N_i of the eigenvalues of the Higgs field classically lie at the critical point a_i of $W(x)$. Quantum effects then have the effect of opening the points a_i into cuts (if $N_i > 0$). The number of $N_i > 0$, *i.e* the number of cuts, is equal to $s = \ell - m$.

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We now to turn to explicit solutions of (2.19). The method we shall adopt is to first find solutions for a $U(p)$ gauge theory and then apply the "multiplication by N/p map" [32], with N/p integer. This will yield a solution for a $U(N)$ gauge group and will allow to take a large N limit with p fixed.

2.2 No double points

We now describe how to engineer the case where the near critical curve (2.14) has no double points, so $m = 0$. This is the situation considered in [3, 33, 34]. In this case, we first consider the consistency conditions (2.19) for a $U(p = n)$ gauge theory with $W(x)$ of order $\ell = n + 1$. In this case, (2.19) are trivially satisfied with

$$W'(x) = N\varepsilon P_n(x)\,, \qquad f_{n-1}(x) = -4N^2\varepsilon^2\Lambda^{2n}\ . \tag{2.22}$$

Notice that with our minimal choice of potential, the on-shell curve actually implies that $S = 0$ since the coefficient of x^{n-1} in $f_{n-1}(x)$ vanishes and so the resolvent falls faster than $1/x$ at infinity. This, of course, is pathological from the point-of-view of the old matrix model and may be remedied by using a non-minimal potential with extra branch points or double points outside the critical region. However, in the holomorphic context in which we are working, having $S = 0$ is perfectly acceptable and we stick with it. The on-shell curve consists of an n-cut hyperelliptic curve and one can verify, by taking the classical limit, that $N_i = 1$, $i = 1, \ldots, n$. The double-scaling limit involves a situation where n branch points, one from each of the cuts, come together. This can be arranged by having

$$W'(x) = N\varepsilon\big(B_n(x) + 2\Lambda^n\big)\,, \qquad B_n(x) = \prod_{j=1}^{n}(x - b_j) \tag{2.23}$$

and then taking the limit (2.15). In this case, the near-critical curve Σ_- (2.16) is of the form

$$y_-^2 = \tilde{B}_n(\tilde{x})\ . \tag{2.24}$$

The important point is that we can tune to the critical region whilst keeping the theory on-shell with respect to the glueball superpotential by simply changing the parameters $\{b_j\}$ which appear in the potential.

Now that we have found a suitable vacuum of a $U(n)$ theory, we now lift this to a $U(N)$ theory with the multiplication by N/n map [32]. Under this map, the $\mathcal{N} = 1$ curve remains intact, including the potential $W(x)$ whilst the Seiberg-Witten curve of the $U(N)$ theory is

$$y_{SW}^2 = P_N(x)^2 - 4\Lambda^{2N} = \Lambda^{2(N-n)}\mathcal{U}_{\frac{N}{n}-1}\Big(\frac{P_n(x)}{2\Lambda^n}\Big)^2\big(P_n(x)^2 - 4\Lambda^{2n}\big)\,, \tag{2.25}$$

where $\mathcal{U}_{\frac{N}{n}-1}(x)$ is a Chebishev polynomial of the second kind. The vacuum of the $U(N)$ theory has $N_i = N/n$, $i = 1, \ldots, n$.

Notice that in the near critical region the Seiberg-Witten curve is identical to Σ_-, up to a rescaling:

$$y_{\rm SW}^2 \longrightarrow \Big(\frac{2N}{n}\Big)^2\Lambda^{2N-n}B_n(x)\ . \tag{2.26}$$

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This is simply a reflection of the observation of [3] that the decoupled sector has enhanced $\mathcal{N} = 2$ supersymmetry. Moreover, if C is a cycle which is vanishing as $a \to 0$ then the integral of the Seiberg-Witten differential around C, which gives the mass of a BPS state carrying electric and magnetic charges in the theory, becomes

$$\oint_C \frac{xP'_N(x)\,dx}{y_{\rm SW}} \longrightarrow -\Lambda^{-n/2} N a^{n/2+1} \oint_C y_-\,d\tilde{x}\ . \tag{2.27}$$

Notice that in the double-scaling limit (2.37) (with $m = 0$) the mass of the state is fixed. This state is a dibaryon that carries electric and magnetic charges of the IR gauge group. In the double-scaling limit, therefore, a set of mutually non-local dibaryons become very light.[4] In fact, the Seiberg-Witten curve at the critical point, $a = 0$, has the form

$$y^2_{\rm SW} = 4\Big(\frac{N}{n}\Big)^2 \Lambda^{2N-n} x^n\,, \tag{2.28}$$

which describes a $\mathbf{Z}_n$ or A_{n-1} Argyres-Douglas singularity [35 – 37].

2.3 With double points

For the case with double points, we cannot simply take two of the branch points $\{b_j\}$ in (2.23) above to be the same. If we simply did that then the zero of the Seiberg-Witten curve, by which we mean a zero of the polynomial H_{N-s} in (2.19), would also be a zero of the $\mathcal{N} = 1$ curve as well. By the analysis of [31], this would imply that the condensate of the associated massless dibaryon is vanishing and the dual $U(1)$ group unconfined. On the contrary, we need to arrange a situation where any zero of the Seiberg-Witten curve is not simultaneously a zero of the $\mathcal{N} = 1$ curve, so that the putative massless dibaryon is condensed and the dual $U(1)$ is confined.

A suitable $\mathcal{N} = 1$ curve which reduces to (2.14) in the near-critical region is

$$y^2 = Z_m(x)^2 B_n(x)\big(B_n(x)H_r(x)^2 + 4\Lambda^{2r+n}\big)\ . \tag{2.29}$$

In this case, we have $\ell = m + n + r$, $s = n + r$ and

$$W'(x) = N\varepsilon Z_m(x)B_n(x)H_r(x)\,, \qquad f_{\ell-1}(x) = 4N^2\varepsilon^2\Lambda^{2r+n} Z_m(x)^2 B_n(x)\ . \tag{2.30}$$

Notice that in order that $f_{\ell-1}(x)$ has order less than ℓ we require $r > m$. The curve (2.29) is actually on-shell with respect to the Seiberg-Witten curve of a $U(2r + n)$ theory with

$$P_{2r+n}(x) = H_r(x)^2 B_n(x) + 2\Lambda^{2r+n}\ . \tag{2.31}$$

In the classical limit, we have two eigenvalues at each of the zeros of $H_r(x)$ and one in each of the zeros of $B_n(x)$. Once again we can employ the multiplication map (2.25) (with n replaced by $2r + n$) to find the vacuum of the $U(N)$ theory we are after.

Notice that the double points of the Seiberg-Witten curve $\{h_i\}$ are not generally zeros of the curve (2.29), which means that the associated dyons are condensed. The near-critical curve Σ_- in this case is

$$y_-^2 = \tilde{Z}_m(\tilde{x})^2 \tilde{B}_n(\tilde{x})\,, \tag{2.32}$$

[4] For $n = 2$ there is only a single light dibaryon.

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while in the near-critical region the Seiberg-Witten curve becomes

$$y_{\rm SW}^2 \longrightarrow 4\Big(\frac{N}{n+2r}\Big)^2 \Lambda^{2N-2r-n} H_r(0)^2 B_n(x) \; . \tag{2.33}$$

where we assumed that the zeros of $H_r(x)$ lie outside the critical region. In this case, the integral of the Seiberg-Witten differential around a vanishing cycle diverges in the double-scaling limit:

$$\oint_C \frac{xP_N'(x)\,dx}{y_{\rm SW}} \sim Na^{n/2+1} = \Delta a^{-m} \to \infty \; . \tag{2.34}$$

So in contrast to the case with no double points, the dibaryon states are very heavy. In addition, the dyon condensate associated to the zero h_i of $H_r(x)$ is given by an exact formula [31]

$$\langle m_i \tilde{m}_i \rangle = N\varepsilon y(h_i) \sim N \to \infty \, , \tag{2.35}$$

where we have assumed that h_i stays fixed as $a \to 0$. So in the double-scaling limit the value of the condensate and hence the confinement scale in the dual $U(1)$, or string tension, occurs at a very high mass scale.

Even though there are no light dibaryons as in the previous example, there is still an interesting double-scaling limit in the gauge theory due to the presence of other light mesonic states in the theory with a mass $\sim \Delta$ [4].

Notice also that contrary to our choice above, if we scale $h_i \to 0$ as $a \to 0$ then the tensions of the confining strings vanish and the theory is at an $\mathcal{N} = 1$ superconformal fixed point in the infra-red corresponding to one of the $\mathcal{N} = 1$ Argyres-Douglas-type singularities described in [33]. As the double points of the Seiberg-Witten curve h_i move away from the origin the associated dyons condense and the superconformal invariance is broken.

2.4 Higher genus

In the $a \to 0$ limit, it was shown in [4] that the genus g free energy gets a dominant contribution from Σ_- of the form

$$F_g \sim \big(Na^{(m+n/2+1)}\big)^{2-2g} \; . \tag{2.36}$$

Note that in this equation N is the one from the field theory and not the matrix model $\hat{N}$. This motivates us to define the double-scaling limit (DSL) [3, 4]

$$a \to 0 \, , \qquad N \to \infty \, , \qquad \Delta \equiv Na^{m+n/2+1} = \text{const} \; . \tag{2.37}$$

Moreover, the most singular terms in a in (2.36) depend only on the near-critical curve (2.16) in a universal way.

It was observed in [15] that eq. (2.36) matches the expected behaviour of the topological B model free energy at the singularity [44]. In fact, as can be seen from (2.14) and (2.15)

$$\Delta \sim N \int y\, dx \; . \tag{2.38}$$

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More precisely, the double-scaling parameter is proportional to the period of the one-form $y\,dx$ on one of the cycles that vanish at the singularity. Moreover, this one-form corresponds to the reduction of the holomorphic 3-form Ω on the underlying Calabi-Yau geometry

$$uv + y^2 = W'(x)^2 + f(x) \tag{2.39a}$$

$$\Omega = \frac{du dv dx}{\sqrt{uv - W'(x)^2 - f(x)}} \,. \tag{2.39b}$$

This comes from the fact that 3-cycles in the Calabi-Yau correspond to $\boldsymbol{S}^2$ fibered over the complex x plane. In particular

$$\int \Omega \sim \int y\,dx \,, \tag{2.40}$$

where Ω is integrated on a vanishing 3-cycle in the Calabi-Yau that reduces to one of the vanishing one-cycles on the matrix mode spectral curve. Putting everything together, we find that

$$F_g \sim \Delta^{2-2g} \sim \left(\int y\,dx\right)^{2-2g} \sim \left(\int \Omega\right)^{2-2g} \tag{2.41}$$

which is precisely the behaviour we expect for the free energy of the topological B model on the Calabi-Yau [44], in agreement with the Dijkgraaf-Vafa correspondence.

2.5 The double-scaling limit of F-terms

In this section, we will review the DSL of various F-terms in the low-energy effective action derived in [3, 4]. These results were used to argue that, in the case of no double points, the supersymmetry of the low-energy theory is actually enhanced from four to eight supercharges. In the next section, we will then compare some of these F-terms with their counterparts in the corresponding $\mathcal{N} = 2$ theory.

The effective action is written in terms of chiral superfields S_l and $w_{\alpha l}$ which are defined as gauge-invariant single-trace operators [29]

$$\begin{aligned} S_l &= -\frac{1}{2\pi i}\oint_{A_l} dx\,\frac{1}{32\pi^2}\mathrm{Tr}_N\left[\frac{W_\alpha W^\alpha}{x-\Phi}\right] \,, \\ w_{\alpha l} &= \frac{1}{2\pi i}\oint_{A_l} dx\,\frac{1}{4\pi}\mathrm{Tr}_N\left[\frac{W_\alpha}{x-\Phi}\right] \,. \end{aligned} \tag{2.42}$$

It will also be convenient to define component fields for each of these superfields,

$$S_l = s_l + \theta_\alpha \chi_l^\alpha + \cdots \,, \qquad w_{\alpha l} = \lambda_{\alpha l} + \theta_\beta f_{\alpha l}^\beta + \cdots \,. \tag{2.43}$$

The component fields, s_l and f_l are bosonic single trace operators whilst χ_l and λ_l are fermionic single trace operators. In the large-N limit, these operators should create bosonic and fermionic colour-singlet single particle states respectively. It is instructive to consider the interaction vertices for these fields contained in the F-term effective action whose general form is given by [11 – 13]

$$\mathcal{L}_F = \mathrm{Im}\left[\int d^2\theta\,\left(W_{\mathrm{gb}} + W_{\mathrm{eff}}^{(2)}\right)\right] \,, \tag{2.44}$$

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where

$$W^{(2)}_{\rm eff} = \frac{1}{2}\sum_{k,l} \frac{\partial^2 F_0}{\partial S_k \partial S_l}\, w_{\alpha k} w^{\alpha}_l \ . \qquad (2.45)$$

Expanding (2.44) in components on-shell, we find terms like

$$\int d^2\theta\, W^{(2)}_{\rm eff} \supset V^{(2)}_{ij} f^i_{\alpha\beta} f^{\alpha\beta j} + V^{(3)}_{ijk} \chi^i_\alpha f^{\alpha\beta j} \lambda^k_\beta + V^{(4)}_{ijkl} \chi^i_\alpha \chi^{\alpha j} \lambda^k_\beta \lambda^{\beta l} \,, \qquad (2.46)$$

where

$$V^{(L)}_{i_1 i_2 \dots i_L} = \frac{\partial^L F_0}{\partial S_{i_1} \partial S_{i_2} \dots \partial S_{i_L}} \qquad (2.47)$$

for $L = 2, 3, 4$. In the large-N limit, $V^{(L)}$ scales like N^{2-L}. We will also consider the 2-point vertex coming from the glueball superpotential

$$\int d^2\theta\, W_{\rm gb} \supset H^{(2)}_{ij} \chi^i_\alpha \chi^{\alpha j} \,, \qquad (2.48)$$

where

$$H^{(2)}_{ij} = \frac{\partial^2 W_{\rm gb}}{\partial S_i \partial S_j} \ . \qquad (2.49)$$

The matrix $H^{(2)}_{ij}$ therefore effectively determines the masses of the chiral multiplets S_l. Note that, in the large-N limit, $H^{(2)}$ scales like N^0.

We begin by considering the couplings $V^{(2)}_{ij}$ of the low-energy $U(1)^s$ gauge group. Each of the $U(1)$'s is associated to one of the glueball fields S_i, or equivalently the set of 1-cycles $\{A_i\}$ on Σ. If we ignore the $U(1)$ associated to the overall 't Hooft coupling S, or the cycle $A_\infty = \sum_{i=1}^s A_i$ which can be pulled off to infinity, the couplings of the remaining ones are simply the elements of the period matrix of Σ.

In order to take the $a \to 0$ limit, it is useful to choose a new basis of 1-cycles $\{\tilde{A}_i, \tilde{B}_i\}$, $i = 1, \dots, s-1$, which is specifically adapted to the factorization $\Sigma \to \Sigma_- \cup \Sigma_+$. The subset of cycles with $i = 1, \dots, [n/2]$ vanish at the critical point while the cycles $i = [n/2]+1, \dots, s-1$ are the remaining cycles which have zero intersection with all the vanishing cycles. If we define the periods on Σ

$$M_{ij} = \oint_{\tilde{B}_j} \frac{x^{i-1}}{\sqrt{\sigma(x)}}\, dx \,, \qquad N_{ij} = \oint_{\tilde{A}_j} \frac{x^{i-1}}{\sqrt{\sigma(x)}}\, dx \,, \qquad (2.50)$$

then the period matrix, in this basis, is simply

$$\Pi = N^{-1} M \ . \qquad (2.51)$$

In appendix B, we calculate the $a \to 0$ limit of these matrices. The results are summarized in (B.4) and (B.7). Using these results, we have

$$\Pi \longrightarrow \begin{pmatrix} N^{-1}_{--} M_{--} & N^{-1}_{--} M^{(0)}_{-+} + \mathcal{N} M^{(0)}_{++} \\ 0 & (N^{(0)}_{++})^{-1} M^{(0)}_{++} \end{pmatrix} \ . \qquad (2.52)$$

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Let us look more closely at the structure of each block in the above matrix. First of all, by (B.2)

$$\left(N_{--}\right)_{ij} \sim a^{n/2-j}\, f^{(N)}_{ij}(\tilde b_l)\,, \qquad \left(M_{--}\right)_{ij} \sim a^{n/2-j}\, f^{(M)}_{ij}(\tilde b_l)\,, \tag{2.53}$$

which implies

$$\left(N_{--}\right)^{-1}_{ij} \left(M_{--}\right)_{jk} = f^{(N)-1}_{ij}(\tilde b_l) f^{(M)}_{jk}(\tilde b_l) = \Pi^{-}_{ik}(\tilde b_l)\ . \tag{2.54}$$

Furthermore, since N_{--}^{-1} vanishes in the limit $a \to 0$, we find that

$$N_{--}^{-1} M^{(0)}_{-+} + \mathcal{N} M^{(0)}_{++} = N_{--}^{-1}\, M^{(0)}_{-+} - N_{--}^{-1}\, N^{(0)}_{-+}\big(N^{(0)}_{++}\big)^{-1} M^{(0)}_{++} \to 0\ . \tag{2.55}$$

Therefore, the period matrix has the following block-diagonal form in the DSL

$$\Pi \longrightarrow \begin{pmatrix} \Pi^- & 0 \\ 0 & \Pi^+ \end{pmatrix}\ . \tag{2.56}$$

The upper block Π^- is actually the period matrix of the near-critical spectral curve Σ_- (2.16) since the cycles $\{\tilde A_i, \tilde B_i\}$, for $i \le [n/2]$ form a standard homology basis for Σ_-. Similarly, the lower block Π^+ is the period matrix of Σ_+. So in the limit $a \to 0$ the curve Σ factorizes as $\Sigma_- \cup \Sigma_+$. The fact that the period matrix factorizes is evidence of the more stringent claim that the whole theory consists of two decoupled sectors $\mathcal{H}_-$ and $\mathcal{H}_+$ in the DSL. Note that although we did not consider it, the $U(1)$ associated to S only couples to the $\mathcal{H}_+$ sector.

We can extend this discussion to include other F-terms that are derived from the glueball superpotential. For example, consider the 3-point vertex

$$V^{(3)}_{ijk} = \frac{\partial^3 F_0}{\partial \tilde S_i \partial \tilde S_j \partial \tilde S_k}\ . \tag{2.57}$$

Here, the $\tilde S_i$ as defined as in (2.6) but with respect to the cycles $\tilde A_i$. They are related to the S_i by an electro-magnetic duality transformation. There is a closed expression for these couplings of the form [41, 42, 39, 43]

$$V^{(3)}_{ijk} = \frac{1}{N\varepsilon} \sum_{l=1}^{2s} \mathrm{Res}_{b_l}\, \frac{\omega_i\, \omega_j\, \omega_k}{dx dy}\ , \tag{2.58}$$

where $\{\omega_j\}$ are the holomorphic 1-forms normalized with respect to the basis $\{\tilde A_i, \tilde B_i\}$. So we can deduce the behaviour of the couplings from our knowledge of the scaling of ω_j. This is derived in appendix B. We find that the couplings are regular as $a \to 0$, except if $i, j, k \le [n/2]$ in which case,

$$V^{(3)}_{ijk} \longrightarrow \left(N\varepsilon a^{m+n/2+1}\right)^{-1} \sum_{l=1}^{n} \mathrm{Res}_{\tilde\sigma_l}\, \frac{\tilde\omega_i\, \tilde\omega_j\, \tilde\omega_k}{d\tilde x dy_-}\,, \tag{2.59}$$

where the $\{\tilde\omega_i\}$ are the one-forms on Σ_-. Therefore, in the DSL proposed in (2.37), we find that these interactions remain finite $\sim \Delta^{-1}$, while the other 3-point vertices $\to 0$. This is yet further evidence of the decoupling of the Hilbert space into two decoupled sectors

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where the interactions in the $\mathcal{H}_-$ sector remain finite in the DSL while those in $\mathcal{H}_+$ go to zero. Notice, also that these interactions of the $\mathcal{H}_-$ sector depend universally on Σ_-.

The final F-term quantity that we consider is the Hessian matrix for the glueball superfields

$$H^{(2)}_{jk} = \frac{\partial^2 W_{\rm gb}}{\partial \tilde{S}_j \partial \tilde{S}_k} \; . \tag{2.60}$$

Using (2.58) we find

$$H^{(2)}_{jk} = \sum_{i=1}^{s} N_i \frac{\partial^3 F_0}{\partial \tilde{S}_i \partial \tilde{S}_j \partial \tilde{S}_k} = \frac{1}{N\varepsilon} \sum_{l=1}^{2s} \mathrm{Res}_{b_l} \frac{T\,\omega_j\,\omega_k}{dx dy} \,, \tag{2.61}$$

where we have defined the 1-form T

$$T = N\varepsilon \sum_{i=1}^{s} \frac{\partial y\, dx}{\partial S_i} \; . \tag{2.62}$$

It is known that T can be can be written simply in terms of the on-shell Seiberg-Witten curve [45]:

$$T = d\log(P_N + y_{\rm SW}) \; . \tag{2.63}$$

In the limit $a \to 0$, we can take the near-critical expressions for $y_{\rm SW}$ in (2.33) and for $P_N(x) = 2\Lambda^N$ to get the behaviour

$$T \longrightarrow \Lambda^{-r-n/2} H_r(0) \frac{N}{n+2r} a^{n/2} d\sqrt{\tilde{B}(\tilde{x})} \sim N a^{n/2} \; . \tag{2.64}$$

We also need

$$dy \longrightarrow a^{m+n/2} d\big(\tilde{Z}_m(x)\sqrt{\tilde{B}(\tilde{x})}\big) \sim a^{m+n/2} \; . \tag{2.65}$$

The scaling of the holomorphic differentials is determined in appendix B.

Counting the powers of N and a, we find that for any j and k, $H^{(2)}_{jk}$ goes like an inverse power of a and hence diverges in the DSL (the powers of N cancel). This, however, presents us with a puzzle. In the case without double points described in [3], the Hessian was shown to vanish for the $\mathcal{H}_-$ sector, *i.e.* $j, k \leq [n/2]$. Let us see how this is compatible with the scaling we have just seen. In the case, $j, k \leq [n/2]$,

$$H^{(2)}_{jk} \sim a^{-(m+1)} \sum_{l=1}^{n} \mathrm{Res}_{\tilde{b}_l} \left[\frac{d\sqrt{\tilde{B}_n(\tilde{x})}\;\tilde{\omega}_j\,\tilde{\omega}_k}{d\tilde{x}\; d\Big(\tilde{Z}_m(\tilde{x})\,\sqrt{\tilde{B}_n(\tilde{x})}\Big)} \right] , \tag{2.66}$$

where

$$\tilde{\omega}_j = \frac{\tilde{L}_j(\tilde{x})}{\sqrt{\tilde{B}_n(\tilde{x})}} d\tilde{x}. \tag{2.67}$$

and $\tilde{L}_j(\tilde{x})$ is a polynomial of degree $[n/2]-1$. Note that the differential $\tilde{\omega}_j\,\tilde{\omega}_k/d\tilde{x}$ has simple poles at $\tilde{x} = \tilde{b}_l$ on the curve Σ_-:

$$\frac{\tilde{\omega}_j\,\tilde{\omega}_k}{d\tilde{x}} = \frac{\tilde{L}_j(\tilde{x})\,\tilde{L}_k(\tilde{x})}{\tilde{B}_n(\tilde{x})} d\tilde{x} \,, \tag{2.68}$$

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but has no pole at $\tilde{x}=\infty$. For example for n odd, we find

$$\frac{\tilde{\omega}_j\,\tilde{\omega}_k}{d\tilde{x}} \longrightarrow \frac{d\tilde{x}}{\tilde{x}^3}\ . \tag{2.69}$$

This means that in the case with no double points, $m=0$, the Hessian matrix elements (2.66) vanish identically:

$$H^{(2)}_{jk} \sim a^{-1}\sum_{l=1}^{n} \mathrm{Res}_{\tilde{b}_l}\left[\frac{\tilde{\omega}_j\,\tilde{\omega}_k}{d\tilde{x}}\right]=0\,, \tag{2.70}$$

because the sum of all residues of a meromorphic differential on the compact near-critical curve Σ_- is identically zero. This is precisely the result found in [3]. On the other hand, if $m>0$, the Hessian matrix element will not vanish in general, because the differential on the right-hand side of (2.66) has extra simple poles at the roots of

$$2\tilde{Z}'_m(\tilde{x})\tilde{B}_n(\tilde{x})+\tilde{Z}_m(\tilde{x})\tilde{B}'_n(\tilde{x})=0\ . \tag{2.71}$$

This result is very significant because it highlights an important difference between the case with and without double points. Even though we do not have control over the kinetic terms of the glueball states, we take this behaviour of the Hessian matrix to signal that, with double points, the masses of the glueball fields become very large in the DSL. This is to be contrasted with the case without double points studied in [3], where the appearance of the $[n/2]$ massless glueballs was interpreted as evidence that supersymmetry is enhanced to $\mathcal{N}=2$ in the DSL.

3. Matrix model loop functions

In [4], it was shown that the large N DSLs defined in [3, 4] map to double-scaling limits of the auxiliary Dijkgraaf-Vafa matrix model, which are completely analogous to the "old" matrix model double-scaling limits [14, 23]. The natural question that arises is which $c\leq 1$ non-critical bosonic strings are dual to these matrix model DSLs [15]?

In general, according to the Dijkgraaf-Vafa correspondence [11 − 13], the matrix model with polynomial superpotential $W_q(\Phi)$ is dual to the topological B model on a non-compact Calabi-Yau geometry which is related to the matrix model spectral curve in a simple way

$$y^2=W'_q(x)^2+f_{q-1}(x) \quad\rightarrow\quad uv+y^2=W'_q(x)^2+f_{q-1}(x)\ . \tag{3.1}$$

The effect of the DSL is to focus in a neighbourhood of a certain singularity of the above family of Calabi-Yau's parametrized by the superpotential couplings and deformation polynomial f_{q-1}. For instance, in the cases considered in [3] where n branch points of the matrix model spectral curve collide, we are in the proximity of a singularity of type A_{n-1}

$$uv+y^2=x^n-\mu\,, \tag{3.2}$$

which is a generalization of the conifold singularity. These non-compact Calabi-Yaus can generically be embedded in weighted projective spaces, for instance (3.2) goes to

$$uv+y^2=x^n-\frac{\mu}{z^k}\,,\quad k=\frac{2n}{n+2} \tag{3.3}$$

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and have been argued to admit a Landau-Ginzburg description with a superpotential determined by the defining equation (3.3) [17, 18] (see also [46, 47]). This is a generalization of the CY/LG correspondence in the compact case. Furthermore, the superstring vacua corresponding to these generic non-compact CYs in the proximity of such singularities are expected to be described by non-critical superstring backgrounds of the form [8, 9]

$$\{SL(2,R)/U(1) \text{ supercoset}\} \times \{\mathcal{N}=2 \text{ minimal model}\} \tag{3.4}$$

and their mirror symmetry partners [48, 49]

$$\{\mathcal{N}=2 \text{ Liouville}\} \times \{\mathcal{N}=2 \text{ minimal model}\} \tag{3.5}$$

More precisely, in [8], it was proposed that the four-dimensional double-scaled Little String Theory with 8 supercharges defined at such singularities has a holographic description in terms of the above non-critical superstring backgrounds. In fact, this correspondence is at the basis of the duality proposal of [3].

Therefore, since the topological B model in a neighbourhood of these CY singularities should be dual to a topological twist of the above non-critical superstring backgrounds, we expect the matrix model DSL to correspond precisely to the topological twist of these non-critical superstring backgrounds. For the singularities (3.2)(3.3), we should consider the A-twist of

$$SL(2,R)_k/U(1) \times SU(2)_n/U(1)\,, \quad k = \frac{2n}{n+2}\,. \tag{3.6}$$

The relation between strings on non-compact Calabi-Yaus and non-critical superstring brackgrounds [18, 8] involving the $N = 2$ Kazama-Suzuki $SL(2)/U(1)$ model or its mirror, $N = 2$ Liouville theory [8, 48, 49], has been studied by several authors (see [50−52] and references therein).

In this section, we will study the matrix model in the DSL and in particular derive exact expressions for its loop functions at genus zero by means of the algorithm developed in [24]. From the loop functions, one can extract correlation functions of operators of the dual $c \leq 1$ theory, as was done for 2d gravity coupled to $(2, 2m-1)$ minimal models [22, 20, 21, 23]. We will then check if the results that can be extracted from the matrix model are compatible with the picture advocated above.

Given the matrix integral

$$Z = \int d\hat{\Phi}\; e^{-\hat{N}\mathrm{Tr}\,V(\hat{\Phi})}\,, \tag{3.7}$$

the p-point loop function is defined as

$$W(x_1,\ldots,x_p) \equiv \hat{N}^{p-2}\Big\langle \mathrm{tr}\frac{1}{x_1-\hat{\Phi}}\cdots\mathrm{tr}\frac{1}{x_p-\hat{\Phi}}\Big\rangle_{\mathrm{conn}} \tag{3.8}$$

and it has the following genus expansion

$$W(x_1,\ldots,x_p) = \sum_{g=0}^{\infty}\frac{1}{\hat{N}^{2g}}W^{(g)}(x_1,\ldots,x_p)\,. \tag{3.9}$$

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The 1-loop function is the Laplace transform of the expectation value of the so-called macroscopic loop operator [20]

$$W(\ell) = \frac{1}{\hat{N}} \mathrm{Tr}\, e^{\ell\,\hat{\Phi}} \tag{3.10}$$

$$W(x) = \int_0^\infty d\ell\, e^{-x\ell}\, \langle\, W(\ell)\,\rangle = \frac{1}{\hat{N}} \langle\, \mathrm{Tr}\, \frac{1}{x-\Phi}\,\rangle \tag{3.11}$$

The macroscopic loop operator $W(\ell)$ corresponds to the insertion of a loop of length ℓ on the two-dimensional discretized matrix model surface and encodes information on local operators in the dual non-critical string [22, 20, 21, 23]. Roughly,

$$W(\ell) \sim \sum_{j\geq 0} \ell^{x_j} \mathcal{O}_j\,, \quad x_j > 0\,, \tag{3.12}$$

where the $\mathcal{O}_j$'s are operators in the $c \leq 1$ system. The correlation functions of the $\mathcal{O}_j$'s can then be extracted by shrinking the macroscopic loops, namely by studying the $\ell \to 0$ limit of $\langle W(\ell_1)W(\ell_2)\rangle$, $\langle W(\ell_1)W(\ell_2)W(\ell_3)\rangle$, *etc.* The n-point correlation functions of the macroscopic loop operators can be found by evaluating the inverse Laplace transform of the matrix model n-loop functions to be described below.

In [24], Eynard found a solution to the matrix model loop equations that allows to write down an expression for the multiloop functions (3.8)(3.9) at any given genus in terms of a special set of Feynman diagrams. The various quantities involved depend only on the spectral curve of the matrix model and in particular one needs to evaluate residues of certain differentials at the branch points of the spectral curve.

This algorithm and its extension to calculate higher genus terms of the matrix model free energy [53] represent major progress in the solution of the matrix model via loop equations [55 − 57, 54]. This is particularly important because, as reviewed in [4], the orthogonal polynomial approach can be applied to multi-cut solutions in very special cases only. Another nice feature of the loop equation algorithm is that it shows directly how the information is encoded in the spectral curve. This fact allowed us make precise statements about the DSL of multiloop functions and higher genus quantities simply by studying the DSL of the spectral curve and its various differentials [4].

We will now give the expression of the 2 and 3-loop functions at genus zero using Eynard's results and then consider their DSL. Given the matrix model spectral curve for an s-cut solution in the form (2.11)

$$y^2 = Z_m(x)^2 \sigma_{2s}(x)\,, \tag{3.13}$$

the genus zero 2-loop function is given by

$$\begin{aligned} W(x_1,x_2) = &-\frac{1}{2(x_1-x_2)^2} + \frac{\sqrt{\sigma(x_1)}}{2\sqrt{\sigma(x_2)}(x_1-x_2)^2} \\ &- \frac{\sigma'(x_1)}{4(x_1-x_2)\sqrt{\sigma(x_1)}\sqrt{\sigma(x_2)}} + \frac{A(x_1,x_2)}{4\sqrt{\sigma(x_1)}\sqrt{\sigma(x_2)}}\,. \end{aligned} \tag{3.14}$$

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The polynomial A is defined as

$$A(x_1,x_2)=\sum_{i=1}^{2s}\frac{\mathcal{L}_i(x_2)\sigma(x_1)}{x_1-\sigma_i}\,, \tag{3.15}$$

where

$$\mathcal{L}_i(x_2)=\sum_{l=0}^{s-2}\mathcal{L}_{i,l}x_2^l=-\sum_{j=1}^{s-1}L_j(x_2)\int_{A_j}\frac{dx}{\sqrt{\sigma(x)}}\frac{1}{(x-\sigma_i)} \tag{3.16}$$

and s is the number of cuts. The order $s-2$ polynomials $L_j(x)$ enter the expression of the holomorphic one-forms ω_j and are fixed by the requirement that these forms are canonically normalized

$$\omega_j=\frac{L_j(x)dx}{\sqrt{\sigma(x)}}\,,\qquad \int_{A_k}\omega_j=\delta_{jk}\,,\qquad j,k=1,\dots,s-1\;. \tag{3.17}$$

The genus zero 2-loop function for coincident arguments is

$$\begin{aligned}W(x_1,x_1)&=\lim_{x_2\to x_1}W(x_1,x_2)=-\frac{\sigma''(x_1)}{8\sigma(x_1)}+\frac{\sigma'(x_1)^2}{16\sigma(x_1)^2}+\frac{A(x_1,x_1)}{4\sigma(x_1)}\\&=\sum_{i=1}^{2s}\frac{1}{16(x-\sigma_i)^2}-\frac{\sigma_i''}{16\sigma_i'(x-\sigma_i)}+\frac{\mathcal{L}_i(x)}{4(x-\sigma_i)}\;.\end{aligned} \tag{3.18}$$

Another important object is the differential

$$dS_{2i-1}(x_1,x_2)=dS_{2i}(x_1,x_2)=\frac{\sqrt{\sigma(x_2)}}{\sqrt{\sigma(x_1)}}\left(\frac{1}{x_1-x_2}-\frac{L_i(x_1)}{\sqrt{\sigma(x_2)}}-\sum_{j=1}^{s-1}C_j(x_2)L_j(x_1)\right)dx_1\,, \tag{3.19}$$

where $i=1,\dots,s$ and

$$C_j(x_2)=\int_{A_j}\frac{dx}{\sqrt{\sigma(x)}}\frac{1}{(x-x_2)}\;. \tag{3.20}$$

A crucial aspect of the one-form (3.19) is that it is analytic in x_2 in the limit $x_2\to\sigma_{2i-1}$ or σ_{2i} [24]

$$\lim_{x_2\to\sigma_i}\frac{dS_i(x_1,x_2)}{\sqrt{\sigma(x_2)}}=\frac{1}{\sqrt{\sigma(x_1)}}\left(\frac{1}{x_1-x_2}-\sum_{j=1}^{s-1}L_j(x_1)\int_{A_j}\frac{dx}{\sqrt{\sigma(x)}}\frac{1}{(x-x_2)}\right)dx_1\;. \tag{3.21}$$

The subtlety is that in the definition of (3.20), the point x_2 is taken to be outside the loop surrounding the j-th cut, whereas in (3.21), x_2 is inside the contour. Note also that

$$A(x_1,x_2)=-\sum_{i=1}^{2s}\left(\sum_{j=1}^{s-1}L_j(x_2)C_j(\sigma_i)\right)\frac{\sigma(x_1)}{x_1-\sigma_i} \tag{3.22}$$

and in particular

$$A(x_1,\sigma_i)=\mathcal{L}_i(x_1)\sigma'(\sigma_i)\;. \tag{3.23}$$

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The expression for the genus zero 3-loop function is

$$W_3(x_1,x_2,x_3) = 2\sum_{i=1}^{2s} \mathrm{Res}_{\sigma_i} W_2(x,x_1)W_2(x,x_2)W_2(x,x_3)\frac{(dx)^2}{dy}$$
$$= \frac{1}{2}\sum_{i=1}^{2s} Z(\sigma_i)^2\,\sigma'(\sigma_i)\;\chi_i^{(1)}(x_1)\chi_i^{(1)}(x_2)\chi_i^{(1)}(x_3) \tag{3.24}$$

where the one-differentials $\chi_i^{(1)}$'s are defined by [15]

$$\chi_i^{(1)}(x_1) = Res_{x\to\sigma_i}\left(\frac{dS_i(x_1,x)}{2y(x)}\,\frac{1}{(x-\sigma_i)}\right)$$
$$= \frac{1}{2Z(\sigma_i)\sqrt{\sigma(x_1)}}\left(\frac{1}{x_1-\sigma_i}+\mathcal{L}_i(x_1)\right)dx_1 \tag{3.25}$$

Incidentally, these expressions reproduce the results for the 2 and 3-loop functions in the one-cut solution given in [21].

3.1 The double-scaling limit

As reviewed in section 2, in the neighbourhood of a singularity where m double points and n branch points of the spectral curve come together

$$y^2 \to CZ_m(x)^2 B_n(x) \tag{3.26}$$

where the double points z_j and the branch points b_i both tend to x_0, which we can take, without loss of generality, to be $x_0 = 0$. The DSL involves first taking $a \to 0$

$$x = a\tilde{x}\,, \qquad z_i = a\tilde{z}_i\,, \qquad b_j = a\tilde{b}_j$$

while keeping tilded quantities fixed. In the limit, we can define the near-critical curve Σ_-:

$$\Sigma_- : \qquad y_-^2 = \tilde{Z}_m(\tilde{x})^2\tilde{B}_n(\tilde{x})\;. \tag{3.27}$$

It was shown in [4] that in the DSL (2.37)

$$a\to 0\,, \qquad N\to\infty\,, \qquad \Delta\equiv Na^{m+n/2+1} = \mathrm{const}$$

The matrix model p-loop functions behave as follows

$$W_p(x_1,\ldots,x_p)\,dx_1\ldots dx_p \;\to\; C^{1-p/2}\,\Delta^{2-p}\,\tilde{W}_p(\tilde{x}_1,\ldots,\tilde{x}_p)\,d\tilde{x}_1\ldots d\tilde{x}_p \tag{3.28}$$

where the tilded quantities are the loop functions corresponding to the near-critical curve Σ_-. This result was derived by considering the limit of all the various differentials and quantities that enter in Eynard's algorithm.

As was emphasized in [3] and in section 2.5, the above results hold if the basis of A and B-cycles on the spectral curve Σ is adapted to the factorization $\Sigma\to\Sigma_+\cup\Sigma_-$. In the case where two branch points collide, this means that one of the A-cycles of this basis shrinks to zero size. Thus, this singularity corresponds to a case where a single cut shrinks, which is equivalent to a conifold singularity. Then we can set

$$y_-^2 = \tilde{\sigma}(\tilde{x}) = \tilde{x}^2 - \tilde{b}^2 \ . \tag{3.29}$$

Dropping the tildes and setting $C = 1$, by (3.14)(3.28), the 2-loop and 3-loop functions become

$$W(x_1, x_2) = \frac{1}{2(x_1 - x_2)^2}\left(\frac{x_1 x_2 - b^2}{\sqrt{x_1^2 - b^2}\sqrt{x_2^2 - b^2}} - 1\right) \tag{3.30}$$

and

$$W_3(x_1, x_2, x_3) = \frac{1}{2\,\Delta}\sum_{i=1}^{2} \sigma'(\sigma_i)\ \chi_i^{(1)}(x_1)\chi_i^{(1)}(x_2)\chi_i^{(1)}(x_3) \tag{3.31}$$

where the one-differentials $\chi_i^{(1)}$'s are defined by

$$\chi_i^{(1)}(x_1) = \frac{1}{2\sqrt{x_1^2 - b^2}}\left(\frac{1}{x_1 - \sigma_i}\right) dx_1 \ . \tag{3.32}$$

The inverse Laplace transform of these genus zero functions can be done explicitly (see appendix A) to find

$$\langle\, W(\ell_1) W(\ell_2)\,\rangle = \sum_{n=1}^{\infty} n\, I_n(b\ell_1)\, I_n(b\ell_2) \tag{3.33}$$

$$\langle\, W(\ell_1) W(\ell_2) W(\ell_3)\,\rangle = \frac{8}{b^2\Delta}\sum_{p,q,r=1}^{\infty} pqr\left(1 + (-1)^{p+q+r}\right) I_p(b\ell_1)\, I_q(b\ell_2)\, I_r(b\ell_3) \tag{3.34}$$

where $I_n(x)$ is the modified Bessel function of the first kind, which satisfies the equation

$$\left[-\left(x\frac{\partial}{\partial x}\right)^2 + x^2 + n^2\right]\psi(x) = 0 \ . \tag{3.35}$$

As we were not able to find a closed analytic expression for the inverse Laplace transform of the double-scaled loop functions for the more complicated singularities, from now on we will restrict our attention to the case of the conifold singularity.

3.2 Comparison between matrix model DSL and $c = 1$ non-critical string at selfdual radius

How do we interpret eqs.(3.33)(3.34)? In [22], a precise dictionary between macroscopic loop operators on the matrix model side and sum of local operators of the dual bosonic non-critical string on the other was proposed. This applied to the matrix model DSLs dual to the $(2, 2m-1)$ minimal models coupled to 2d gravity. The main tool to establish such dictionary was the Wheeler-DeWitt equation satisfied both by the macroscopic loop amplitudes and by the wavefunctions of the non-critical string operators in the so-called "conformal background". In the following, we will apply the same philosophy as [22] starting from eq.(3.33).

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First of all, eq. (3.12) is not exact. In fact from the expression of the matrix model resolvent or 1-loop function in the DSL, one can find that the series on the r.h.s contains terms that diverge in the $\ell \to 0$ limit. This also happened in the cases studied in [22] (and reviewed in [23]). These terms are due to small area divergences, and their coefficients are analytic in the coupling constants. They are interpreted as non-universal contributions and are not relevant in the comparison between the matrix model and the non-critical string. These divergences in the $\ell \to 0$ limit are expected to disappear for higher loop functions. In fact, the 2 and 3-point macroscopic loop correlators, eqs.(3.33) and (3.34), do not show any divergence in the limit $\ell \to 0$, as was the case for the models considered in [22].

The crucial point [22] is that wavefunctions of local non-critical operators are expected to satisfy the Wheeler-DeWitt equation for the $c \leq 1$ system coupled to 2d gravity in the minisuperspace approximation, where only the zero mode ϕ_0 of the Liouville field is taken into account, $\ell \to e^{\gamma\phi_0/2}$

$$\left(- \left(\ell \frac{\partial}{\partial \ell} \right)^2 + 4\mu^2 \ell^2 + \nu^2 \right) \psi_{\mathcal{O}}(\ell) = 0 \ . \tag{3.36}$$

The coefficient ν^2 is related to the conformal dimension $\Delta^0(\mathcal{O})$ of the undressed matter operator by

$$\nu^2 = \frac{8}{\gamma^2} \left[\frac{Q^2}{8} - (1 - \Delta^0(\mathcal{O})) \right] = \frac{4}{\gamma^2} \left(\alpha - \frac{Q}{2} \right)^2 \tag{3.37}$$

and α is the Liouville charge associated to the dressing operator $e^{\alpha\phi}$. The Liouville background charge Q and the exponent γ are given by

$$Q = \frac{2}{\gamma} + \gamma \, , \quad \gamma = \frac{1}{\sqrt{12}} \left(\sqrt{25 - c_M} - \sqrt{1 - c_M} \right) \, , \tag{3.38}$$

where c_M is the central charge of the matter sector. The Liouville central charge c_L is

$$c_L = 1 + 3Q^2 \, , \quad c_L + c_M = 26 \ .$$

Solutions of (3.36) are linear combinations of $I_\nu(2\mu\ell)$ and $K_\nu(2\mu\ell)$, the modified Bessel functions of the first and second kind respectively.

Comparison with (3.33) suggests a change of basis from the operators $\{\mathcal{O}_i\}$ considered in (3.12) to a new basis $\{\sigma_n\}$

$$W(\ell) = \sum_{n=1}^{\infty} \ell^n \mathcal{O}_n = \sum_{n=1}^{\infty} n \frac{I_n(b\ell)}{b^n} \, \sigma_n \ . \tag{3.39}$$

Instead of using $\ell, \ell^2, \ldots$ as a basis to expand the macroscopic loop operator, we can use the basis $I_1, I_2, \ldots$. Since as $\ell \to 0$, $I_n(b\ell) \sim (b\ell)^n$, the change of basis is upper triangular and furthermore the coefficients of this map are analytic functions of b^2. The wavefunctions of the operators σ_n are given by shrinking one of the loops

$$\psi_n(\ell) \ = \ \langle \, W(\ell) \, \sigma_n \, \rangle \ . \tag{3.40}$$

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From (3.33) and (3.39) we find

$$\psi_n(\ell) \;=\; \langle\, W(\ell)\, \sigma_n \,\rangle \;=\; b^n I_n(b\ell)\ . \tag{3.41}$$

This wavefunction satisfies the differential equation

$$\left(-\left(\ell\frac{\partial}{\partial\ell}\right)^2 + 4\mu^2\ell^2 + n^2 \right)\psi_n(\ell) = 0\,, \quad b = 2\mu\ . \tag{3.42}$$

It is remarkable that in the case of the conifold singularity the wavefunctions extracted from the matrix model, eq. (3.41), satisfy the Wheeler-DeWitt equation derived in the minisuperspace approximation exactly, eq. (3.36), as the cases studied in [22]. This may not be true for the more complicated singularities studied in [3], (eq. (3.2) for $n > 2$).

However, contrary to the case of minimal model coupled to 2d gravity studied in [22], the wavefunctions (3.41) are concentrated in the region $\ell \sim e^{\gamma\phi_0/2} \gg 1$ whereas they vanish in the $\ell \to 0$ limit. On the other hand, in [58, 59], it was argued that if a wavefunction is to correspond to a physical operator in the dual $c \le 1$ non-critical bosonic string then it should have support in the region $\ell \ll 1$, which corresponds to infinitesimally small worldsheet metrics $e^{\gamma\phi_0}|dz|^2$. Thus, it should behave like a modified Bessel function of the second kind, $K_\nu(2\mu\ell)$. We can then conclude that the operators σ_n do not correspond to local physical observables because they do not satisfy this requirement. The Liouville operator that dresses them will not satisfy the Seiberg bound $\alpha \le \frac{Q}{2}$ [58, 59].

Nevertheless, we will see that this is actually not a contradiction. As explained at the beginning of section 3 , we expect the matrix model DSL to be dual to the A-twist of the $N = 2$ supersymmetric coset $SL(2)_k/U(1)$ at level $k = 1$, the topological cigar, which was shown in [16] to be equivalent to the $c = 1$ system at selfdual radius (see also [25–27] for a recent analysis). This equivalence was later explained in [17], which showed the relation with the topological theory at a conifold singularity. By the Dijkgraaf-Vafa correspondence [11–13, 19], we then expect the matrix model with near-critical spectral curve (3.29), which is relevant for the conifold singularity, to capture the $c = 1$ theory at selfdual radius.

The key result is that we can actually identify the operators σ_n with a subset of the operators in the $c = 1$ theory at selfdual radius. In the notation of [60], we find that

$$\sigma_n \;\to\; Y^-_{\frac{n}{2},\frac{n}{2}} = c\bar{c}\, e^{i\sqrt{2}nX_0/2} e^{\sqrt{2}(1+n/2)\phi}\,, \quad n = 1, 2, \dots \tag{3.43}$$

This identification is due to the following observations. Setting $c_M = 1$ in (3.38) we find

$$\gamma = \sqrt{2}\,, \quad Q = 2\sqrt{2}\,, \tag{3.44}$$

and from the Wheeler-DeWitt equation (3.42)

$$\nu^2 = \frac{4}{\gamma^2}\left(\alpha - \frac{Q}{2}\right)^2 = 2\left(\alpha - \sqrt{2}\right)^2 = n^2\,, \quad n = 1, 2, \dots \tag{3.45}$$

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Observe that ν^2 is invariant under $\alpha \to Q-\alpha$. This corresponds to the fact that the conformal dimension of the Liouville operator $e^{\alpha\phi}$

$$\Delta(\alpha) = \frac{1}{2}\alpha(Q-\alpha) \tag{3.46}$$

is invariant under the reflection $\alpha \to Q-\alpha$. But, we also know from (3.41) that the wave-function of the Liouville operator is not concentrated in the region $\ell << 1$ and therefore the corresponding α does not satisfy the Seiberg bound $\alpha \leq \frac{Q}{2} = \sqrt{2}$. The solutions to (3.45) compatible with this condition are

$$\alpha_n = \sqrt{2} + \frac{n}{\sqrt{2}}, \quad n = 1, 2, \ldots \tag{3.47}$$

in agreement with eq. (3.43). This is to be contrasted with the solutions

$$\tilde{\alpha}_n = Q - \alpha_n = \sqrt{2} - \frac{n}{\sqrt{2}}, \quad n = 1, 2, \ldots$$

The operators (3.43) indeed correspond to a subset of the full observables in the topological cigar which is given by

$$\begin{aligned} Y^+_{\frac{n}{2},-\frac{n}{2}} &= c\bar{c}\, e^{-i\sqrt{2}nX_0/2} e^{\sqrt{2}(1-n/2)\phi}, \quad n = 0, 1, 2, \ldots \\ Y^-_{\frac{n}{2},\frac{n}{2}} &= c\bar{c}\, e^{i\sqrt{2}nX_0/2} e^{\sqrt{2}(1+n/2)\phi}, \quad n = 0, 1, 2, \ldots \end{aligned} \tag{3.48}$$

and their duals [16, 25, 26]

$$\begin{aligned} Y^+_{\frac{n}{2},\frac{n}{2}} &= c\bar{c}\, e^{i\sqrt{2}nX_0/2} e^{\sqrt{2}(1-n/2)\phi}, \quad n = 0, 1, 2, \ldots \\ Y^-_{\frac{n}{2},-\frac{n}{2}} &= c\bar{c}\, e^{-i\sqrt{2}nX_0/2} e^{\sqrt{2}(1+n/2)\phi}, \quad n = 0, 1, 2, \ldots \end{aligned} \tag{3.49}$$

As we see, the full spectrum also contains operators that satisfy the Seiberg bound.

In summary, after performing the change of basis (3.39), we were able to show that the matrix model DSL corresponding to the conifold singularity, eq. (3.29), describes a subset of the full spectrum of the $c = 1$ non-critical bosonic theory at selfdual radius. In particular, it contains "wrong-branch" operators, namely operators that do not satisfy the Seiberg bound. Therefore, at least for the case of the conifold singularity, we see an explicit relation between the DSL of the Dijkgraaf-Vafa matrix model and the topologically twisted non-critical superstring backgrounds (3.6).

4. Comparison with $\mathcal{N} = 2$ Seiberg-Witten theory

In this section we will further study and discuss the enhancement to $\mathcal{N} = 2$ supersymmetry of the $\mathcal{N} = 1$ effective action for the case with no double points. In particular, we will compare the double-scaling limit (2.37) of the third derivatives of the matrix model free energy

$$\frac{\partial^3 F_0}{\partial S_i \partial S_j \partial S_k} \tag{4.1}$$

with

$$\frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_j \partial a_k}\,, \tag{4.2}$$

where $\mathcal{F}$ is the prepotential of an $\mathcal{N}=2$ pure $SU(n)$ Seiberg-Witten theory in the neighbourhood of an A_{n-1} Argyres-Douglas superconformal fixed point. It is understood that S_i, a_i, $i=1,\dots,[(n-1)/2]$, are the periods of the matrix model and Seiberg-Witten differentials around the cycles in the critical region. This means that we have chosen the same basis of one-cycles on the spectral and Seiberg-Witten curves as described in section 2.5.

The goal is to provide a further consistency check that in the double-scaling limit (2.37) the F-terms of the large N theory are equivalent to those of an $\mathcal{N}=2$ Seiberg-Witten model in the neighbourhood of an Argyres-Douglas singularity. We will exploit exact identities that relate the third derivatives of the genus zero matrix model free energy and the Seiberg-Witten prepotential to a sum of residues on the spectral curve and Seiberg-Witten curve [39, 38]

$$\frac{\partial^3 F_0}{\partial S_i \partial S_j \partial S_k} = \sum_{l=1}^{2n} \mathrm{Res}_{\sigma_l}\left(\frac{\omega_i\omega_j\omega_k}{dx dy}\right) \tag{4.3a}$$

$$\frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_j \partial a_k} = \sum_{l=1}^{2n} \mathrm{Res}_{s_l}\left(\frac{\hat\omega_i\hat\omega_j\hat\omega_k}{dx\, T_{SW}}\right) \tag{4.3b}$$

$$y_{SW}^2 = P_n(x)^2 - 4\Lambda^{2n}\,, \quad T_{SW} \equiv \frac{P_n' dx}{y_{SW}} = d\log(P_n + y_{SW}) \tag{4.3c}$$

As before, the $\omega_i, \hat\omega_j$'s are canonically normalized holomorphic one-differentials on the matrix model and Seiberg-Witten curves

$$\frac{\partial y dx}{\partial S_i} = \omega_i\,, \qquad \frac{\partial \lambda_{SW}}{\partial a_j} = \hat\omega_j\ . \tag{4.4}$$

and σ_l, s_l are the zeroes of these curves. These formulae and their generalizations were used in [38, 39] to derive a set of WDVV-like equations in Seiberg-Witten [40] and Dijkgraaf-Vafa theories.

For the particular matrix model singularities we are interested in, where n branch points collide and there are no double-points, the relevant matrix model spectral curve (2.22)

$$y^2 = \frac{1}{(N\varepsilon)^2}\big(W'(x)^2 + f_{\ell-1}(x)\big) = P_n(x)^2 - 4\Lambda^{2n} \tag{4.5}$$

coincides with the Seiberg-Witten curve of an $SU(n)$ theory

$$y_{SW}^2 = P_n(x)^2 - 4\Lambda^{2n} \tag{4.6}$$

where

$$P_n(x) \to \frac{1}{N\varepsilon} W_n'(x) \tag{4.7}$$

The crucial fact is that, in the double-scaling limit (2.37)

$$T_{SW} \to a^{n/2} dy_-\,, \qquad dy \to a^{n/2} dy_-\,, \tag{4.8}$$

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so that

$$\frac{\partial^3 F_0}{\partial S_i \partial S_j \partial S_k} \to \left(N\varepsilon\, a^{n/2+1}\right)^{-1} \sum_{l=1}^{n} \mathrm{Res}_{\tilde{\sigma}_l} \left(\frac{\tilde{\omega}_i \tilde{\omega}_j \tilde{\omega}_k}{d\tilde{x} dy_-}\right) ,$$

$$\frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_j \partial a_k} \to \left(a^{n/2+1}\right)^{-1} \sum_{l=1}^{n} \mathrm{Res}_{\tilde{\sigma}_l} \left(\frac{\tilde{\omega}_i \tilde{\omega}_j \tilde{\omega}_k}{d\tilde{x} dy_-}\right) , \tag{4.9}$$

where the $\tilde{\omega}_i$'s, $i = 1, \ldots, [(n-1)/2]$ are holomorphic differentials on the near-critical spectral curve

$$\Sigma_- : \quad y_-^2 = \tilde{B}_n(\tilde{x}) , \tag{4.10}$$

and the $\tilde{\sigma}_l$'s are the n zeroes of the polynomial $\tilde{B}_n(\tilde{x})$. We see that the third derivatives in (4.9) have exactly the same dependence on the near-critical spectral curve.

This relation between the double-scaling limit of a Dijkgraaf-Vafa matrix model and relative gauge theory defined at a singularity where n branch points collide and an $\mathcal{N} = 2$ Seiberg-Witten theory with gauge group $SU(n)$ in the proximity of the analogous A_{n-1} Argyres-Douglas singularity, which was shown to hold at genus zero, should extend to the higher genus F-terms as well. In particular, if the Dijkgraaf-Vafa correspondence holds beyond the planar limit, the double-scaling limit of the higher genus terms of the matrix model free energy should be related to the higher genus Seiberg-Witten prepotentials of an $SU(n)$ theory close to an A_{n-1} Argyres-Douglas superconformal fixed point.

This correspondence also makes contact with the work of Nekrasov [61], where it was conjectured that the full Seiberg-Witten partition function is actually the tau-function of a KP hierarchy and that it is related to the theory of a chiral boson living on the Seiberg-Witten curve. In general, a matrix model partition function is the partition function of a chiral boson living on the matrix model spectral curve itself and is a tau-function of the KP hierarchy [62, 19].

It is shown in [4] that, in the double-scaling limit, the higher genus terms of the matrix model free energy with spectral curve Σ behave as follows

$$F_g(\Sigma) \quad \to \quad \left(N\varepsilon\, a^{n/2+1}\right)^{2-2g} F_g(\Sigma_-) , \tag{4.11}$$

where Σ_- is the near-critical spectral curve (4.10) and $F_g(\Sigma_-)$ is the related genus g matrix model free energy which can be evaluated by means of the algorithms developed in [24, 53]. Then the correspondence between the double-scaled matrix model and the $\mathcal{N} = 2$ Seiberg-Witten theory in the neighbourhood of an Argyres-Douglas singularity would imply that

$$\mathcal{F}_g(\Sigma) \quad \to \quad \left(a^{n/2+1}\right)^{2-2g} F_g(\Sigma_-) . \tag{4.12}$$

where $F_g(\Sigma_-)$ is again the genus g matrix model free energy associated to the near-critical spectral curve as in (4.11).

Based on these arguments, the Seiberg-Witten partition function in the proximity of an Argyres-Douglas singularity would be related to the theory of a chiral boson living on the near-critical spectral curve Σ_-.

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5. Discussion

The analysis performed in this paper is a preliminary step towards showing that the matrix model double-scaling limits (DSLs) introduced in [3] and further studied in [4] are dual to the topological twist of the non-critical superstring backgrounds (3.6)

$$SL(2,R)_k/U(1) \times SU(2)_n/U(1)\,, \quad k = \frac{2n}{n+2}\,. \tag{5.1}$$

These non-critical superstring backgrounds are dual to double-scaled Little String Theories in four dimensions [8] and to the large N double-scaling limit of certain $\mathcal{N}=1$ $SU(N)$ gauge theories in a partially confining phase [3]. Actually, the matrix model in question is precisely the matrix model that allows to evaluate the F-terms of these four-dimensional $\mathcal{N}=1$ theories, following the seminal work of Dijkgraaf and Vafa [11 − 13].

Using the solution of the loop equations provided by Eynard for a general matrix model multicut solution [24], we have found the expression of 2 and 3-loop matrix functions in the DSL. From these, one can in principle extract the dictionary between matrix model macroscopic loop operators and operators of the dual $c \leq 1$ non-critical bosonic string, as was done in [22] for the $(2, 2m-1)$ minimal models coupled to 2d gravity.

In the simplest case, where the matrix model DSL is associated to a conifold singularity ($n = 2$ in (5.1)), we have shown explicitly that the spectrum and wavefunctions of the operators of the non-critical bosonic string that can be extracted from the matrix model macroscopic loop correlators are a subset of the full spectrum of the A-twist of the $SL(2)/U(1)$ supercoset at level $k = 1$ or equivalently the $c = 1$ non-critical bosonic string at selfdual radius [16 − 18].

An outstanding problem is to determine the ground ring of the twisted non-critical superstring background (3.6) and see the geometry (3.2) emerge from the ring relations as was done in the $c = 1$ case. The results of [27, 50] would be particularly useful in this respect. One could then couple the analysis of the ground ring with the study of the topological branes of (3.6) and essentially derive the matrix model dual, as was done in [63 − 65] for minimal string theories. Finally, it would also be interesting to study the relation between the matrix model DSL and a topological Landau-Ginzburg model generalizing the analysis carried out in [46, 47] for the conifold/$c = 1$ case.

We have also carried out another check that the F-terms of the large N double-scaled theories considered in [3] are equivalent to those of an $\mathcal{N}=2$ Seiberg-Witten model in the neighbourhood of an Argyres-Douglas singularity. This also suggests that the all-genus Seiberg-Witten partition function in a neighbourhood of such singularities is equivalent to the double-scaled matrix model partition function corresponding to the near-critical spectral curve (4.10), which makes contact with the work of Nekrasov [61].

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A. The macroscopic loop correlators for the topological cigar

In this appendix, we will evaluate the inverse Laplace trasform of the double-scaled 2 and 3-loop functions (3.30)(3.31) and derive eqs.(3.33)(3.34).

The inverse Laplace transfom of the 2-loop function is given by the double Bromwich integral

$$\langle W(\ell_1)W(\ell_2)\rangle = \frac{1}{(2\pi i)^2}\int\int \frac{1}{2(x_1-x_2)^2}\left(\frac{x_1x_2-b^2}{\sqrt{x_1^2-b^2}\sqrt{x_2^2-b^2}}-1\right)e^{\ell_1x_1+\ell_2x_2}dx_1dx_2$$

A generic 2-loop genus zero function has no pole at $x_1 = x_2$, (3.18), and the same is true for the above integrand. Therefore, we can deform the contours of integration and we find

$$\langle W(\ell_1)W(\ell_2)\rangle = \frac{1}{(2\pi i)^2}\int_A\int_A \frac{1}{2(x_1-x_2)^2}\left(\frac{x_1x_2-b^2}{\sqrt{x_1^2-b^2}\sqrt{x_2^2-b^2}}-1\right)e^{\ell_1x_1+\ell_2x_2}dx_1dx_2$$

where A is the loop that surrounds the cut $[-b, b]$ in both the x_1 and x_2 planes. With the change of variables

$$x_i = \frac{b}{2}\left(t_i + \frac{1}{t_i}\right)$$

$$\langle W(\ell_1)W(\ell_2)\rangle = \frac{1}{(2\pi i)^2}\int_{\gamma_0}\int_{\gamma_0}\frac{1}{(1-t_1t_2)^2}e^{\frac{b}{2}\ell_1(t_1+1/t_1)+\frac{b}{2}\ell_2(t_2+1/t_2)}dt_1dt_2$$

where γ_0 is a counterclockwise loop around $t_i = 0$. By the identities

$$e^{\frac{b}{2}\ell_i(t_i+1/t_i)} = \sum_{n=-\infty}^{\infty} I_n(b\ell_i)\, t_i^n$$

$$\frac{1}{(1-t_1t_2)^2} = \sum_{m=1}^{\infty} m(t_1t_2)^{m-1}$$

where $I_n(x)$ is the modified Bessel function, we find (3.33)

$$\langle W(\ell_1)W(\ell_2)\rangle = \sum_{n=1}^{\infty} n\, I_{-n}(b\ell_1)\, I_{-n}(b\ell_2) = \sum_{n=1}^{\infty} n\, I_n(b\ell_1)\, I_n(b\ell_2)\ . \tag{A.1}$$

As for the 3-loop function we can proceed in a similar manner. First of all

$$\chi_b^{(1)}(x_i) = \frac{1}{2\sqrt{x_i^2-b^2}}\left(\frac{1}{x_i-b}\right)dx_i = \frac{2dt_i}{b(1-t_i)^2}$$

$$\chi_{-b}^{(1)}(x_i) = \frac{1}{2\sqrt{x_i^2-b^2}}\left(\frac{1}{x_i+b}\right)dx_i = \frac{2dt_i}{b(1+t_i)^2}$$

Since

$$W_3(x_1,x_2,x_3) = \frac{1}{2\,\Delta}\left(\sigma'(b)\ \chi_b^{(1)}(x_1)\chi_b^{(1)}(x_2)\chi_b^{(1)}(x_3) + \sigma'(-b)\ \chi_{-b}^{(1)}(x_1)\chi_{-b}^{(1)}(x_2)\chi_{-b}^{(1)}(x_3)\right)$$

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we find

$$\langle W(\ell_1)W(\ell_2)W(\ell_3)\rangle =$$

$$\frac{8}{b^2\Delta}\frac{1}{(2\pi i)^3}\int_{\gamma_0}\int_{\gamma_0}\int_{\gamma_0}\left(\prod_{i=1}^{3}\frac{dt_i}{(1-t_i)^2}e^{\ell_i\frac{b}{2}(t_i+1/t_i)}-\prod_{i=1}^{3}\frac{dt_i}{(1+t_i)^2}e^{\ell_i\frac{b}{2}(t_i+1/t_i)}\right)$$

$$=\frac{8}{b^2\Delta}\sum_{p,q,r=1}^{\infty}pqr\left(1+(-1)^{p+q+r}\right)I_p(b\ell_1)\,I_q(b\ell_2)\,I_r(b\ell_3)\ . \tag{A.2}$$

B. Details of the double-scaling limit

In this appendix, we consider the DSL of various quantities defined on the curve Σ (2.11). This is most conveniently done in the basis $\{\tilde{A}_i,\tilde{B}_i\}$ of 1-cycles described in section 2.5. In particular, for $i\leq[n/2]$ these are cycles on the near-critical curve Σ_- in the DSL.

The key quantities that we will need are the periods

$$M_{ij}=\oint_{\tilde{B}_j}\frac{x^{i-1}}{\sqrt{\sigma(x)}}\,dx\,,\qquad N_{ij}=\oint_{\tilde{A}_j}\frac{x^{i-1}}{\sqrt{\sigma(x)}}\,dx\ . \tag{B.1}$$

First of all, let us focus on N_{ij} where $j\leq[n/2]$, but i arbitrary. By a simple scaling argument, as $a\to 0$,

$$N_{ij}=\int_{b^-_{(j)}}^{b^+_{(j)}}\frac{x^{i-1}}{\sqrt{B(x)}}\,dx\longrightarrow a^{i-n/2}\int_{\tilde{b}^-_{(j)}}^{\tilde{b}^+_{(j)}}\frac{\tilde{x}^{i-1}}{\sqrt{\tilde{B}(\tilde{x})}}\,d\tilde{x}=a^{i-n/2}\,f^{(N)}_{ij}(\tilde{b}_l)\,, \tag{B.2}$$

for some function $f^{(N)}_{ij}$ of the branch points of Σ_-. Here, $b^{\pm}_{(j)}$ are the two branch points enclosed by the cycle $\tilde{A}_j$. A similar argument shows that M_{ij} scales in the same way:

$$M_{ij}\longrightarrow a^{i-n/2}\,f^{(M)}_{ij}(\tilde{b}_l)\ . \tag{B.3}$$

So both N_{ij} and M_{ij}, for $i,j,\leq[n/2]$, diverge in the limit $a\to 0$. On the contrary, by using a similar argument, it is not difficult to see that, for $j>[n/2]$, N_{ij} and M_{ij} are analytic as $a\to 0$ since the integrals are over non-vanishing cycles.

In summary, in the limit $a\to 0$, the matrices N and M will have the following block structure

$$N\longrightarrow\begin{pmatrix}N_{--} & N^{(0)}_{-+}\\ 0 & N^{(0)}_{++}\end{pmatrix}\,,\qquad M\longrightarrow\begin{pmatrix}M_{--} & M^{(0)}_{-+}\\ 0 & M^{(0)}_{++}\end{pmatrix}\,, \tag{B.4}$$

where by $-$ or $+$ we denote indices in the ranges $\{1,\dots,[n/2]\}$ and $\{[n/2]+1,\dots,s-1\}$ respectively. In (B.4), N_{--} and M_{--} are divergent while the remaining quantities are finite as $a\to 0$.

We also need the inverse $L=N^{-1}$. In the text, we use the polynomials $L_j(x)=\sum_{k=1}^{s-1}L_{jk}x^{k-1}$, which enter the expression of the holomorphic 1-forms associated to our basis of 1-cycles,

$$\oint_{\tilde{A}_i}\omega_j=\delta_{ij}\ . \tag{B.5}$$

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These 1-forms are equal to

$$\omega_j(x) = \frac{L_j(x)}{\sqrt{\sigma(x)}}\,dx \ = \frac{\sum_{k=1}^{s-1} L_{jk} x^{k-1}}{\sqrt{\sigma(x)}}\,dx\,, \qquad \oint_{A_i} \omega_j(x) = \delta_{ij} \tag{B.6}$$

where $i, j = 1, \ldots, s-1$. From the behaviour of N in the limit $a \to 0$, we have

$$L = N^{-1} \longrightarrow \begin{pmatrix} N_{--}^{-1} & \mathcal{N} \\ 0 & \left(N_{++}^{(0)}\right)^{-1} \end{pmatrix}, \qquad \mathcal{N} = -N_{--}^{-1}\, N_{-+}^{(0)} \left(N_{++}^{(0)}\right)^{-1}. \tag{B.7}$$

Since N_{--} is singular we see that L is block diagonal in the limit $a \to 0$. This is just an expression of the fact that the curve factorizes $\Sigma \to \Sigma_- \cup \Sigma_+$ as $a \to 0$. In this limit, using the scaling of elements of L_{jk}, we find, for $j \leq [n/2]$,

$$\omega_j \longrightarrow \frac{\sum_{k=1}^{[n/2]} (f^{(N)})_{jk}^{-1} \tilde{x}^{k-1}}{\sqrt{\tilde{B}(\tilde{x})}} d\tilde{x} = \tilde{\omega}_j\ . \tag{B.8}$$

the holomorphic 1-forms of Σ_-. While for $j > [n/2]$,

$$\omega_j \longrightarrow \frac{\sum_{k>[n/2]}^{s-1} (N_{++}^{(0)})_{jk}^{-1} x^{k-n/2-1}}{\sqrt{F(x)}} dx\,, \tag{B.9}$$

are the holomorphic 1-forms of Σ_+.

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